Goniometric Diagram Mapping for Hemisphere

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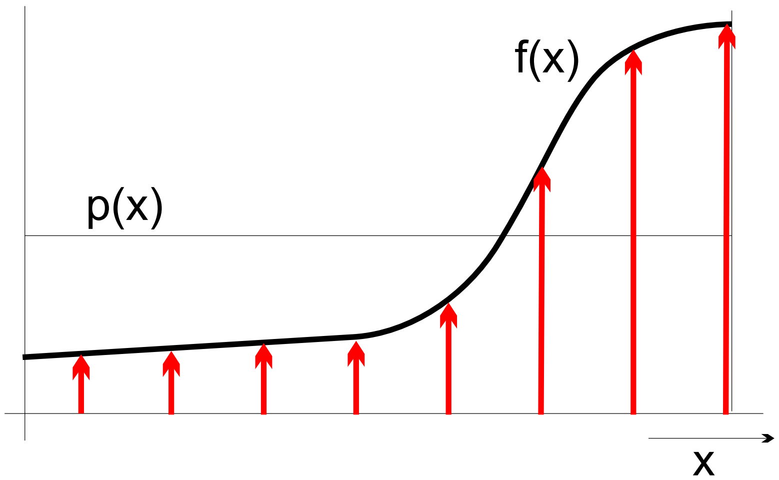
- **Introduction: importance sampling, goniometric diagram.**
- **Algorithm outline.**
- **Detailed explanation of mapping phases.**
- **Results, future work, and applications.**

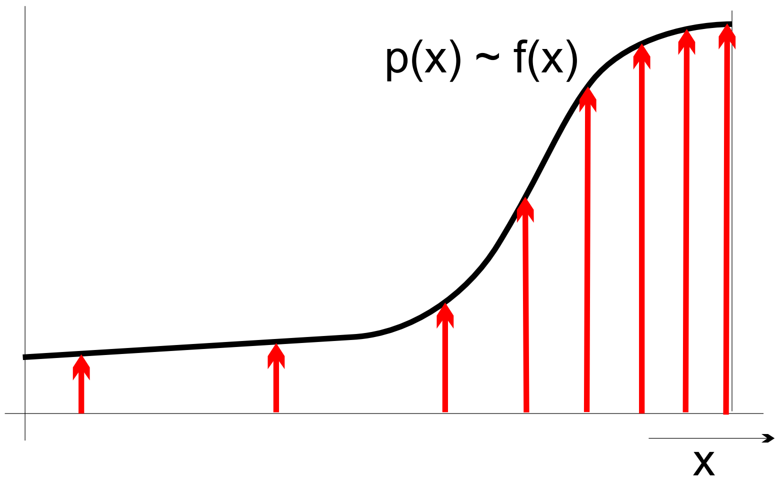
Introduction - PDF

Probability Density Function (PDF)

PDF properties for continuous random variable $p(x)$:

- $p(X) \geq 0 \quad \forall X$
- $\int p(X) = 1$
- Estimator for integration of unknown $f(x)$: $\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$
- Uniform sampling versus Importance sampling





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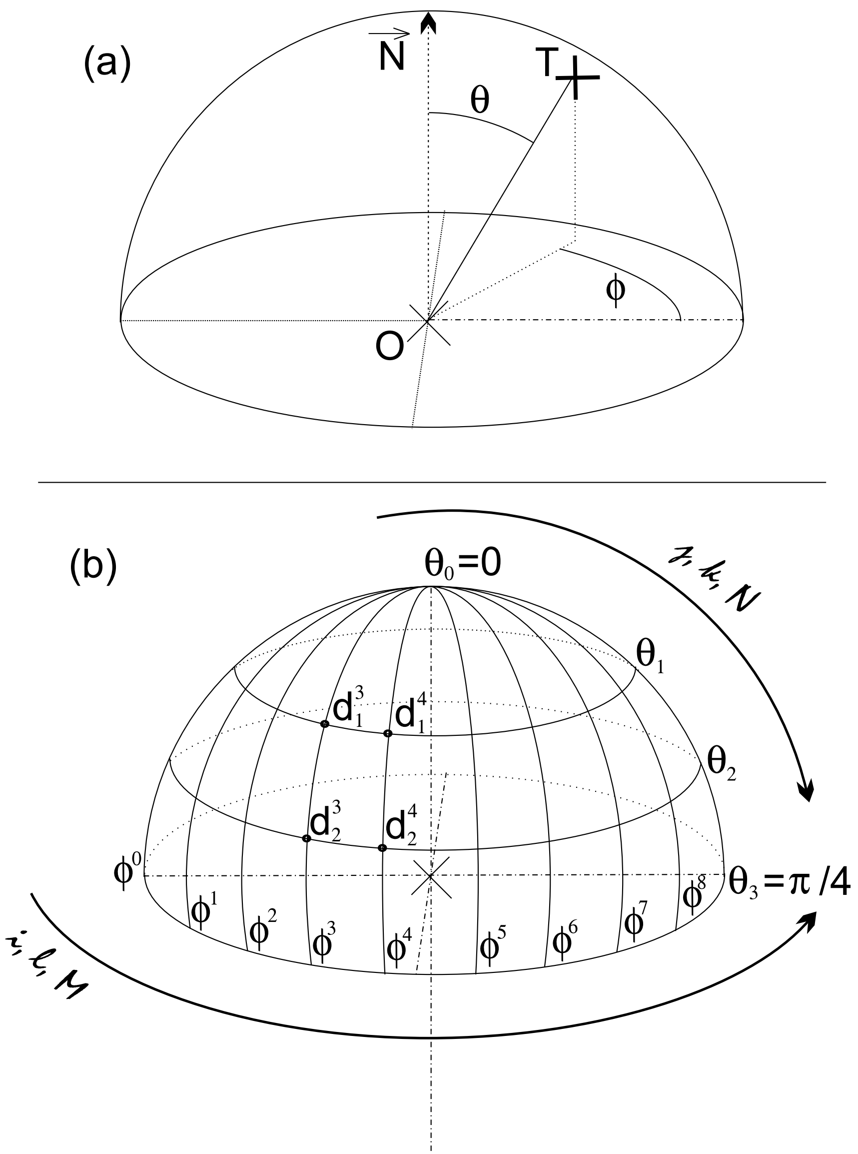
Introduction - Goniometric Diagram

Goniometric Diagram for Hemisphere

- N parallels ($\phi = const$) and M meridians ($\theta = const$)
- PDF given at key points.

Mapping

- $\vec{Y} = f(X), \quad X \in R^2$. (from unit square to hemisphere).
- bijective (unique).
- bicontinuous.
- with fast importance sampling.



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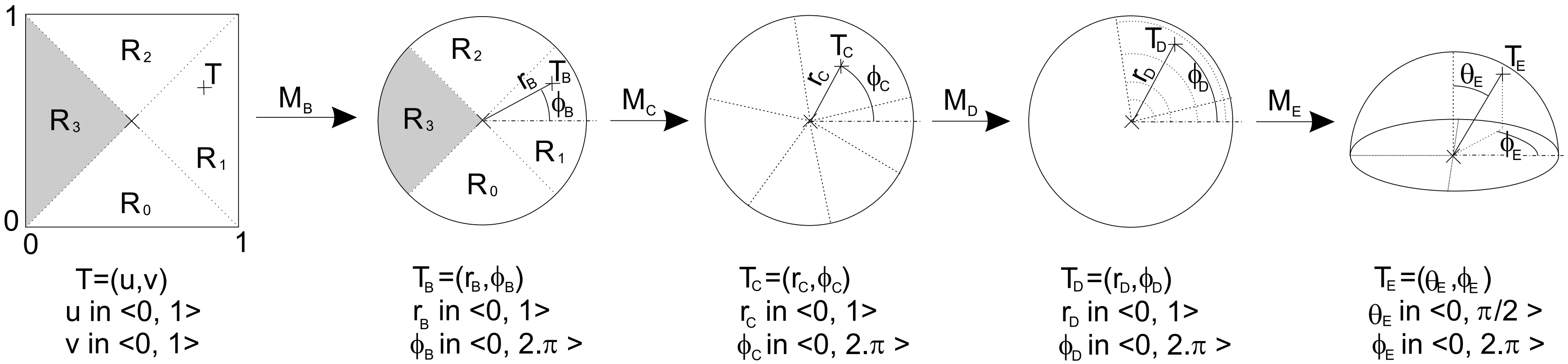
Algorithm Outline

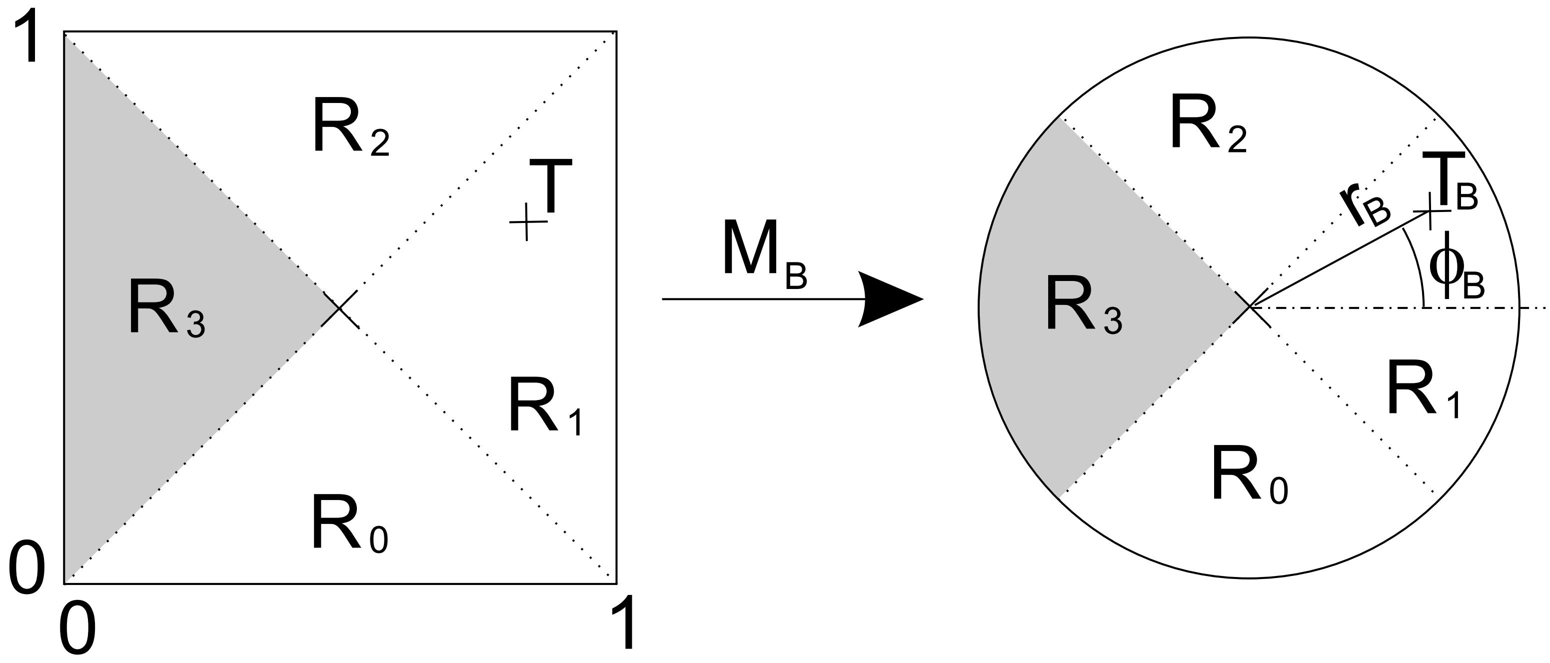
New Composed Mapping



Properties of the proposed "algorithmic mapping".

- Mapping M_B and M_E introduced in CG by Shirley, 1992.
- Mapping M_C and M_D is the new - solving the integral equation on the fly.
- Whole mapping is approximative with guaranteed error.

Mapping M_B



$T=(u,v)$
u in <0, 1>
v in <0, 1>

$T_B=(r_B,\phi_B)$
r_B in <0, 1>
ϕ_B in <0, 2.π >

Mapping M_B continued

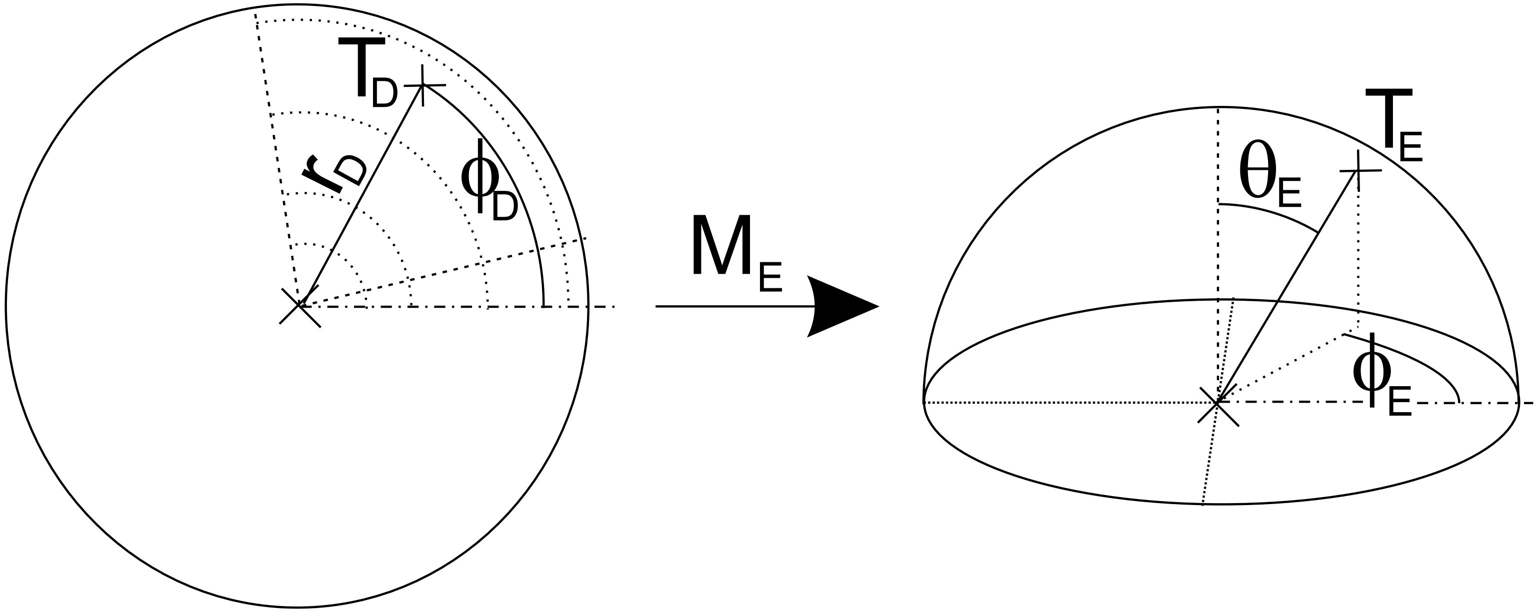
Detailed algorithm in JGT'97, Vol.2, No. 3, page 45-52. by Peter Shirley and Kenneth Chiu

Basic Principle and Properties

- Determine the quadrant of a point X and remap to a disk sector.
- Preserves the adjacency and fractional area.
- Exhibits low distortion.
- Original algorithm modified to provide ϕ in range $\phi \in \langle 0, 2 \cdot \pi \rangle$

Mapping M_E

In CG originally introduced by Peter Shirley, 1992



$T_D = (r_D, \phi_D)$

r_D in $<0, 1>$

ϕ_D in $<0, 2.\pi>$

$T_E = (\theta_E, \phi_E)$

θ_E in $<0, \pi/2>$

ϕ_E in $<0, 2.\pi>$

1 2 3 4 5 6 **7** 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

Mapping M_E continued

Properties

- Density from 2D to 3D is multiplied by constant 1/2.
- Preserves the adjacency and fractional area.
- Does not preserve linearity of mapping from radius to angle!

Formulas

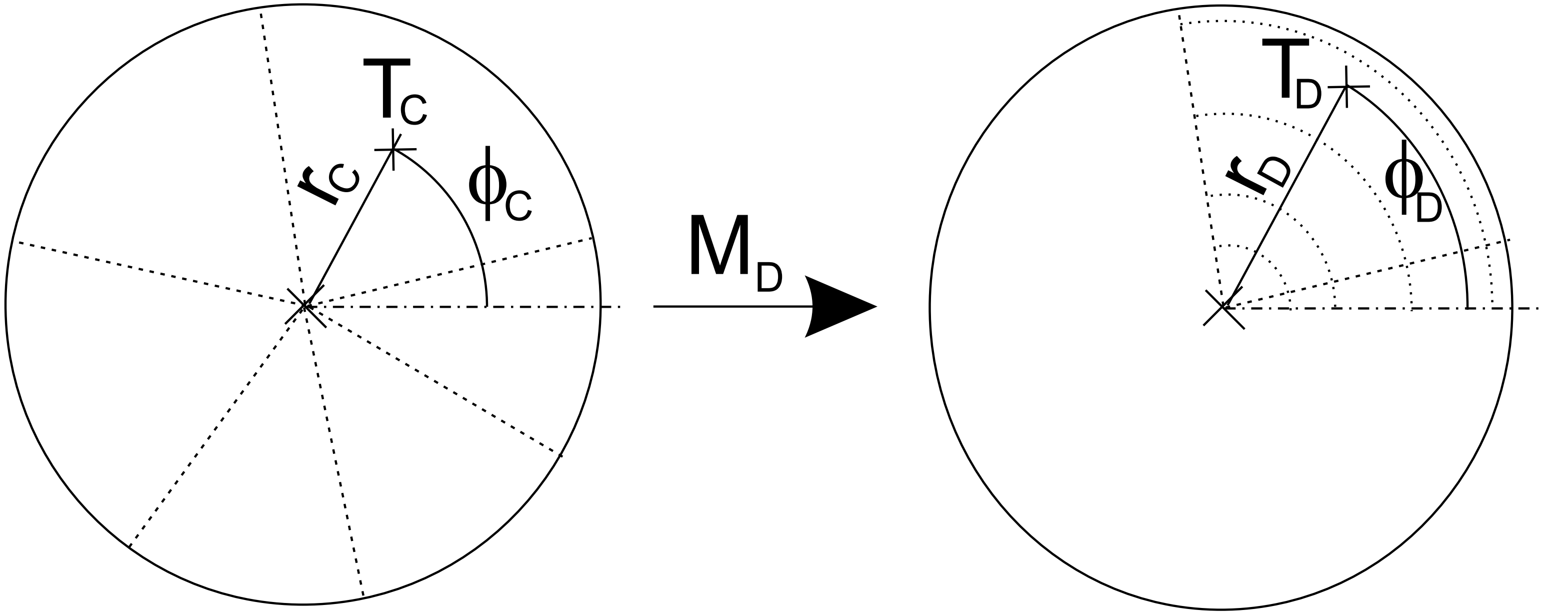
$$
\begin{aligned}
x &= u.\sqrt{2-u^2-v^2} \\
y &= v.\sqrt{2-u^2-v^2} \\
z &= 1-u^2-v^2 \\
u,v &\in \langle -1,1\rangle \times \langle -1,1\rangle
\end{aligned}
$$

or

$$
\begin{aligned}
\phi_E &= \phi_D \\
\theta_E &= \arccos(1-r_D^2) \\
r_D &\in \langle 0,1\rangle
\end{aligned}
$$

Mapping M_D

is solving the integral equation on the fly + binary search for parallels.



$T_C=(r_C,\phi_C)$

r_C in $<0, 1>$

ϕ_C in $<0, 2.\pi>$

$T_D=(r_D,\phi_D)$

r_D in $<0, 1>$

ϕ_D in $<0, 2.\pi>$

Mapping M_D continued

- $\phi_D = \phi_C$ (inside one sector on a 2D disk).
- radius r_D computed from radius r_C, solving the integral equations for piece-wise linear PDF function $d(x)$:

$$d(x) = (d_{j+1} - d_j) \cdot \frac{x - x_j}{x_{j+1} - x_j} + d_j, \quad x \in \langle x_j, x_{j+1} \rangle$$

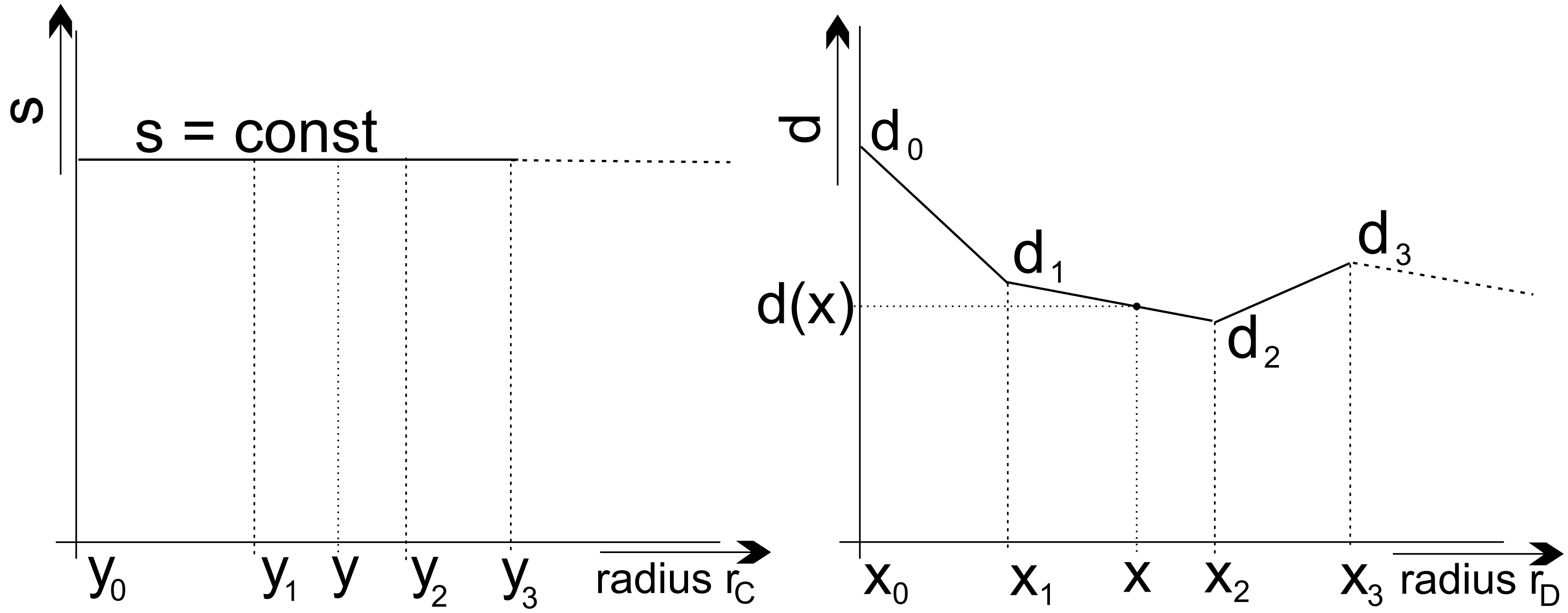
$$\int_{x=x_j}^{X} d(x) \cdot (2 \cdot \pi \cdot x) \cdot dx = \int_{y=y_j}^{Y} s \cdot (2 \cdot \pi \cdot y) \cdot dy$$

- For precomputation given x_j and x_{j+1} we need:

$$\int_{x=x_j}^{x_{j+1}} d(x) \cdot (2 \cdot \pi \cdot x) \cdot dx = \int_{y=y_j}^{y_{j+1}} s \cdot (2 \cdot \pi \cdot y) \cdot dy$$

Mapping M_D continued

- Case $\phi = const$:



Initial condition: $x_0 = y_0 = 0$ (center of the disk).

Mapping M_D continued

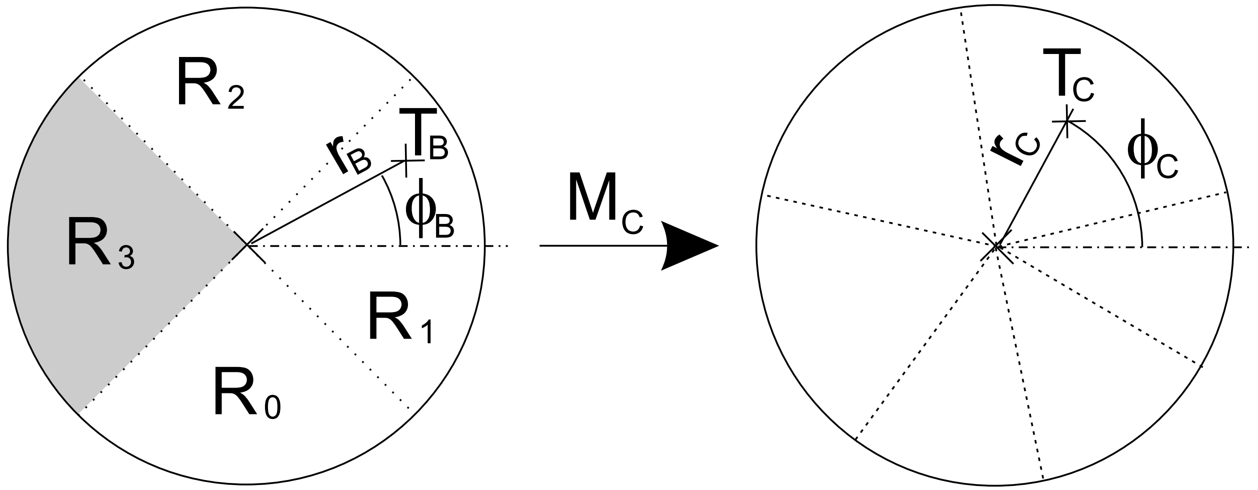
- Integral equation leads to cubic equation with respect to x_{j+1}:

$$
\begin{aligned}
(y_{j+1})^2 \quad = \quad & \frac{d_{j+1}}{s} \cdot (x_{j+1}^2 - x_j^2) + \\
& \frac{d_j - d_{j+1}}{s \cdot (x_{j+1} - x_j)} \cdot (\frac{1}{3} x_{j+1}^3 - x_{j+1} \cdot x_j^2 + \frac{2}{3} \cdot x_j^3) + y_j^2
\end{aligned}
$$

- Cardan's formulae are used for cubic equation above (analytical solution).
- The solution in square form can be linearly interpolated for neighboring ϕ angles !
- Precomputation phase requires to store the $N \times M$ values.
- Binary search $\mathcal{O}(\log N)$ is used for N parallels.

Mapping M_C

- Both radius r and angle ϕ are changed.
- Intuition: Necessary to avoid rejection sampling in radius direction, input is constant density, output has required density.



$T_B = (r_B, \phi_B)$
r_B in $<0, 1>$
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$T_C = (r_C, \phi_C)$
r_C in $<0, 1>$
ϕ_C in $<0, 2.\pi >$

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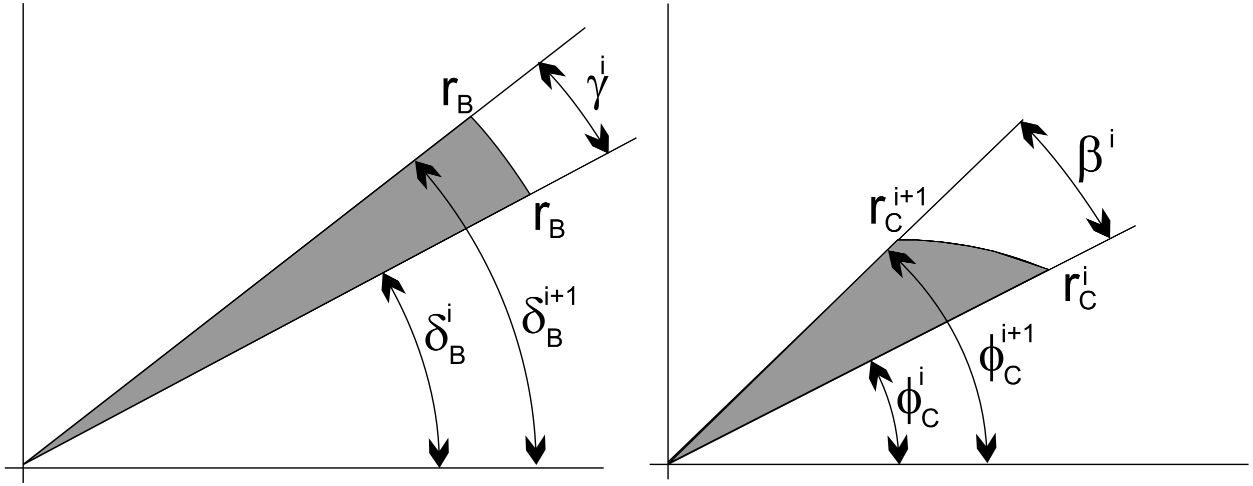
Mapping M_C continued

- Mapping corresponds to solving (another) integral equation.



$$\int_{\phi=0}^{\beta^i} \int_{r=0}^{r_u} r \cdot d\phi \cdot dr = \int_{\phi=0}^{\gamma^i} \int_{r=0}^{\sqrt{\frac{1}{s^{i_{\max}}} \cdot U(i_{\max}, N-1)}} r \cdot d\phi \cdot dr,$$

where

$$r_u = \sqrt{\left(1 - \frac{\phi}{\beta^i}\right) \cdot \frac{1}{s^i} \cdot U(i, N-1) + \frac{\phi}{\beta^i} \cdot \frac{1}{s^{i+1}} \cdot U(i+1, N-1)}.$$

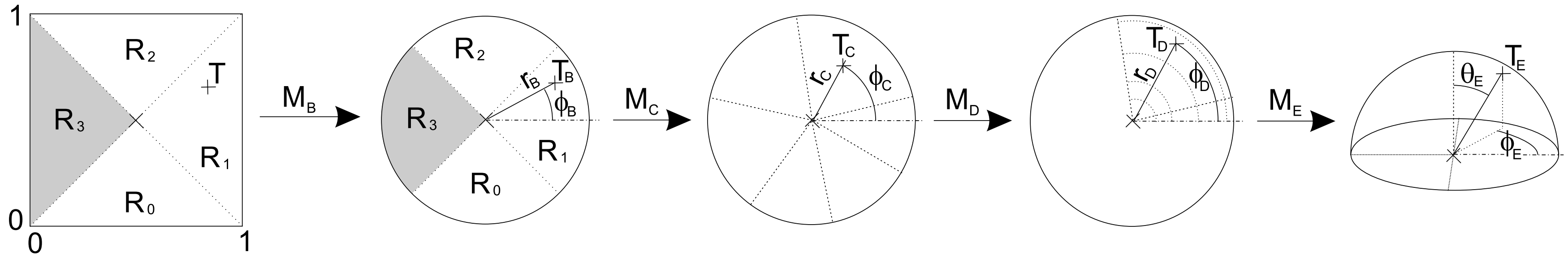
Mapping M_C continued

- Solving integral equation results in a quadratic equation with single root with respect to the unknown variable ϵ:

$$(\epsilon)^2 \cdot [\frac{\beta^i}{2} \cdot (\frac{1}{s^{i+1}} \cdot U(i+1, N-1) - \frac{1}{s^i} \cdot U(i, N-1))] +$$
$$\epsilon \cdot [\frac{\beta^i}{s^i} \cdot U(i, N-1)] - \gamma(\phi_B) \cdot \frac{1}{s^{i_{\max}}} \cdot U(i_{\max}, N-1) = 0$$

- In precomputation we solve the equation for input values, resulting in M precomputed values for M meridians.
- During the mapping, we perform binary search with the time complexity $\mathcal{O}(\log M)$.
- Mapping property: the power over a sector of the disc remains constant before and after mapping M_C !

Whole Mapping Review



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r_C in <0, 1>
ϕ_C in <0, 2.π >

$T_D=(r_D,\phi_D)$
r_D in <0, 1>
ϕ_D in <0, 2.π >

$T_E=(\theta_E,\phi_E)$
θ_E in <0, π/2 >
ϕ_E in <0, 2.π >

Properties of approximative hemispherical mapping:

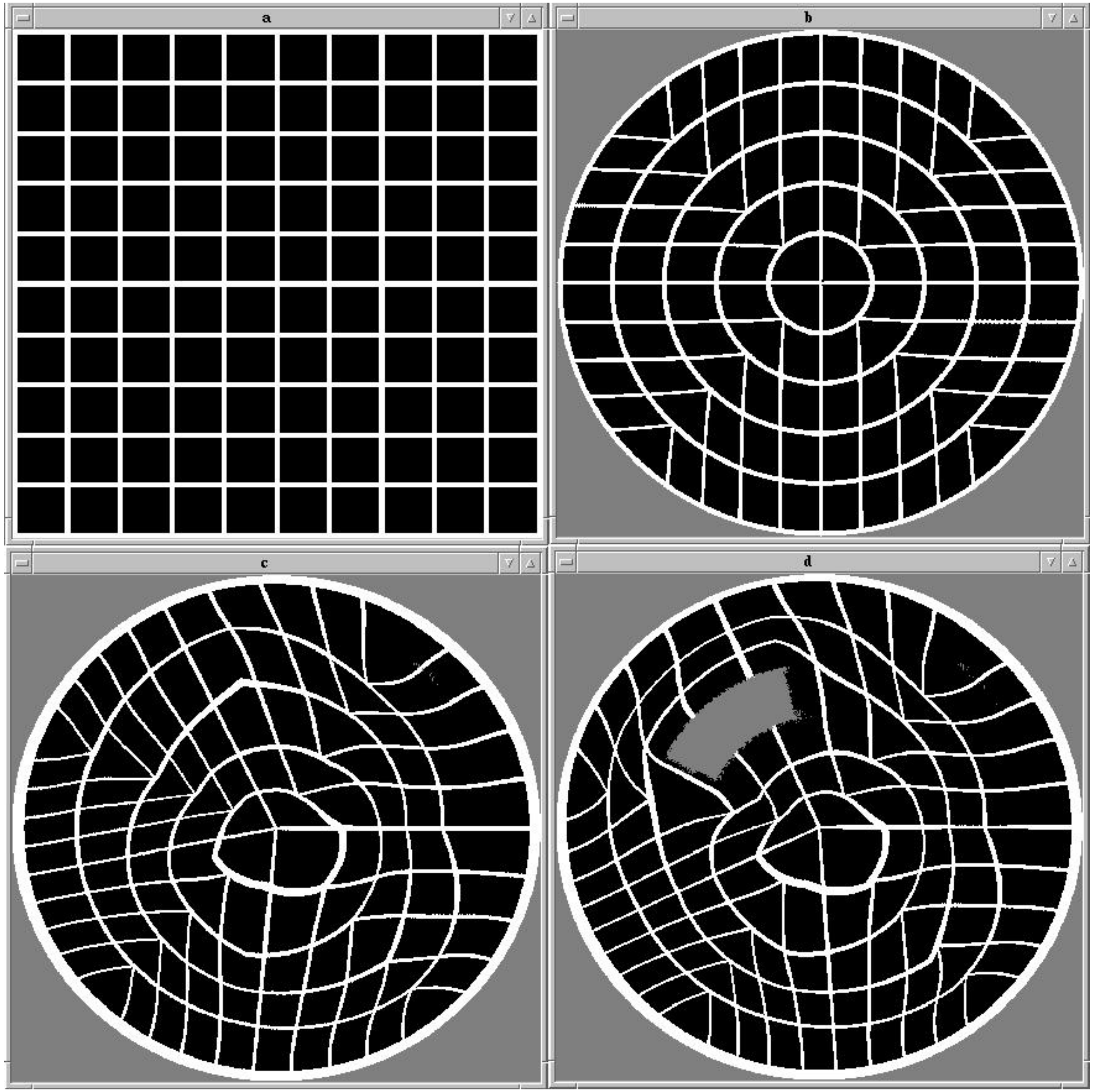
- Bicontinuous, bijective (unique) mapping from a unit square to a hemisphere, where PDF is given by a goniometric diagram.
- Mappings M_B and M_E: low distortion, preserves fractional area, modify the density multiplicatively by constants.
- Mappings M_C and M_D: solving the integral equation analytically on the fly,
  requiring a binary search with time complexity $\mathcal{O}(\log M + \log N)$

Implementation and Results

Properties:

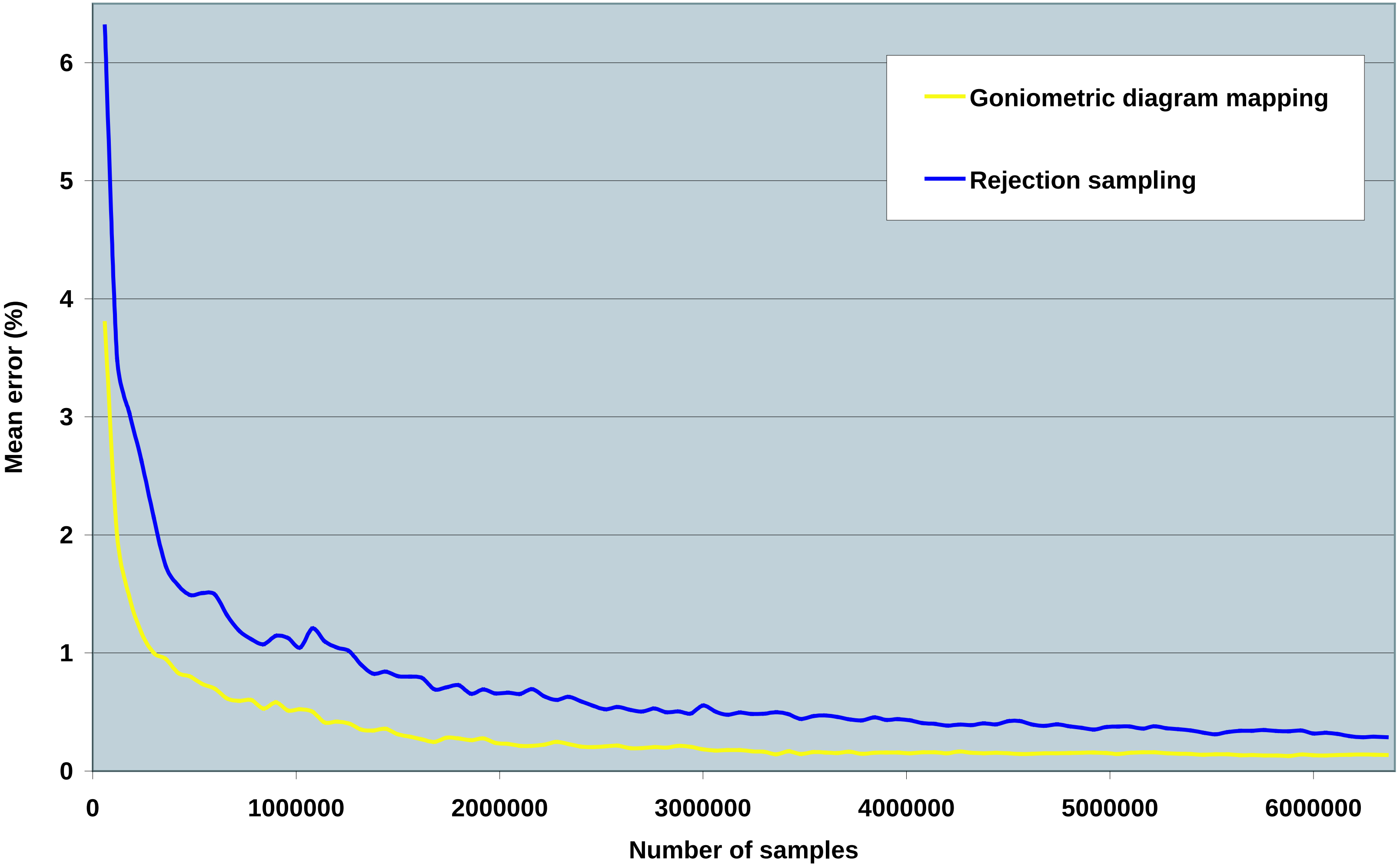
- To our best knowledge, the first algorithm for this problem - it cannot be compared to rejection sampling, since no samples are rejected!
- Memory cost for the representation: $M.N + 2.M + 2$ floating point values. Original PDF requires $M.N + N$ floating point values.
- Time complexity is $\mathcal{O}(\log M + \log N)$.
- Guaranteed distance to exact PDF. It can be compensated by intensity of samples in the range -20 to +25%, however for typical goniometric diagram as small as -4 to +4%.
- Speed on PC, Intel 2.6 GHz is about 332,000 mapped samples per second, including QMC Halton sampling, base 2 and 3.

Results – continued



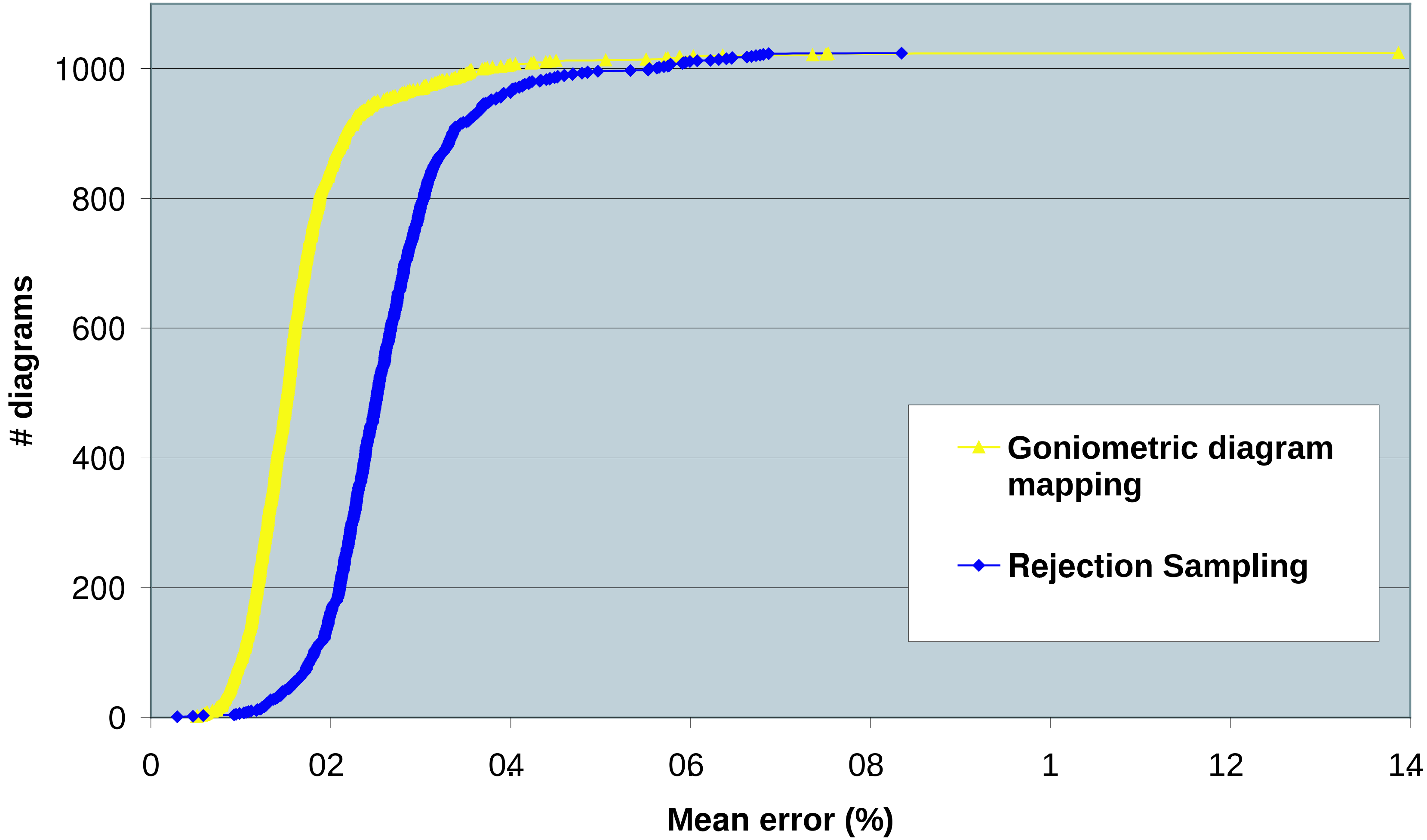
Results – continued

Comparison with rejection sampling.

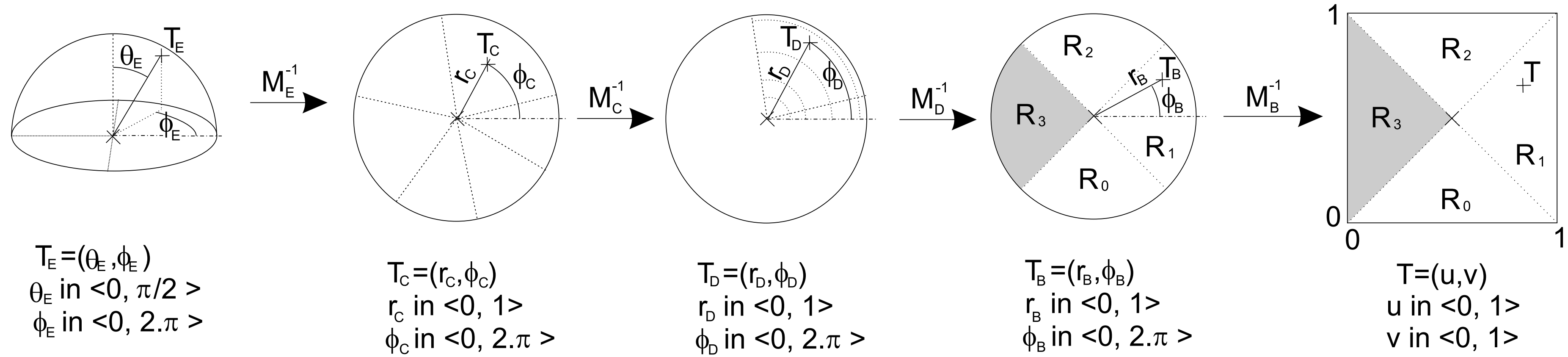


Results – continued

Results for 1024 tested goniometric diagrams.



Results - Inverse Mapping

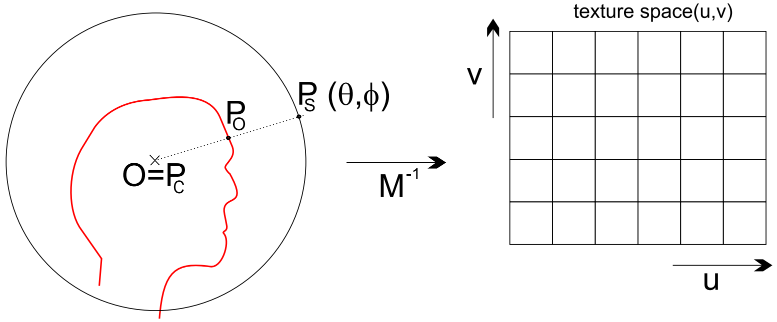


Properties:

- Time complexity again $\mathcal{O}(\log M + \log N)$
- Much simpler analytic formulas.
- Speed on PC, Intel 2.6 GHz is about 6,700,000 mapped samples per second, with QMC Halton sampling, base 2 and 3, about 20 times faster than for forward mapping!

Future Work and Applications

- Native - Importance sampling of point/area light sources described by goniometric diagram.
- Bidirectional Reflectance Distribution Function (BRDF) - importance sampling for isotropic and anisotropic tabulated BRDFs.
- Extension from hemisphere to the full sphere.
- For inverse mapping - texture mapping of weakly convex objects



- Hopefully some others - principle is general.

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Acknowledgements

belongs to

- Karol Myskowski and Jaroslav Křivánek for their remarks.
- Cyrille Damez for proofreading the paper.
- RealReflect project IST-2001-34744 for partial support.
- Peter Shirley for original algorithm with constant density.
- Anonymous reviewers, EG’2003 full papers, for their remarks.

QUESTIONS ?

- NOW - Do not worry!
- Look into the paper. Maybe read twice. Sorry about the math.
- E-mail to me: Vlastimil Havran
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  E-mail: `HAVRAN@MPI-SB.MPG.DE`

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